Communication Network Design: Balancing Modularity and Mixing via Extremal Graph Spectra

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Abstract

By leveraging information technologies, organizations now have the ability to design their communication networks and crowdsourcing platforms to achieve different performance goals. However, research on network *design* has not incorporated any notion of teams, which are known to have many performance benefits in problem-solving networks, instead focusing only on designing networks for fast diffusion of information. Here, we fill this gap by providing a design framework and methodology that incorporates both modularity and mixing time. We take advantage of prior literature in the area of spectral graph theory and demonstrate how desirable aspects of organizational structure can be mapped parsimoniously onto the spectrum of the graph Laplacian derived from a matrix representation of that communication structure. We rely on recent advances in convex optimization to extremize our objective, defined in terms of elements of the graph spectra. Finally, we also present and discuss the resulting communications structures that balance modularity and mixing time.

1 Introduction

The network structure of communication patterns is known to have major consequences for individual and organizational performance, and considerable research has been undertaken to understand those consequences (Bavelas 1950, McCubbins et al. 2009, Mason and Watts 2012, March 1991, Huang and Cummings 2011, Cummings and Cross 2003, Mason et al. 2008, McEvily and Marcus 2005, Sparrowe et al. 2001, Suri and Watts 2011, Lazer and Friedman 2007, Aral and Van Alstyne 2011, Bae et al. 2011, Borgatti and Cross 2003, Capaldo 2007, Reagans and Zuckerman 2001, Sundararajan et al. 2012, Kearns 2012). By leveraging modern information and communications technology, there is now the opportunity for organizations to go beyond understanding their internal network structure and actually design the network structure of their organizational communication networks or crowdsourcing platforms to achieve different performance outcomes.

Unfortunately, however, there is little theory to inform the design of networks of communicating human beings. Rather, the design literature has focused on problems of minimal or optimally "efficient" networks, with applications in non-human infrastructure networks (Balakrishnan et al. 1989, Magnanti and Wong 1984, Minoux 1989, Guimerà et al. 2002, Donetti et al. 2005, Dionne and Florian 1979, Winter 1987, Kershenbaum et al. 1991, Lubotzky et al. 1988, Estrada 2007). Lovejoy and Sinha (2010) is a notable exception in that it is concerned with social networks within organizations, but it is similar in its orientation toward efficiency and short paths between any given individuals in the network. There is indeed substantial theoretical justification for pursuing short paths as a design criterion in human as well as infrastructural networks that is generally understood in terms of two related ideas: that weak-ties enable rapid diffusion of information (Watts and Strogatz, 1998) and that bridging structural holes can be associated with innovation (Burt, 2004).

Although these are important issues, there are also advantages to modularity – having separate but internally cohesive clusters in organizations – but this has to our knowledge been omitted as a network design criterion. Within organizations, internally cohesive groups tend to use similar language constructs, which enables high-bandwidth communication (Aral and Van Alstyne 2011) and increases their effectiveness (Hansen 1999, Reagans and Zuckerman 2001). Shore et al. (2013) show experimentally that for problem solving requiring extensive search of information space or coordination, clustering is beneficial. Additionally, certain types of information and behaviors spread more easily within rather than between clusters (Centola 2010). Finally, real organizations are usually structured in divisions, work groups, or teams — lending an added importance to incorporating some notion of modularity into network design work. Despite all of this, the design literature has yet to address situations in which modularity is desirable.

Two major issues may have stood in the way of incorporating modularity into design work. First, obtaining modularity and short path lengths imply quite different network structures, making theoretical analysis that encompasses both properties difficult. Second, the space of all possible networks is combinatorially large, making the design problem formidably complex (for example, the number of possible undirected graphs with 16 nodes is 2^{120}, or approximately 1.3×10^{36} — far too many to evaluate individually by any known means). Here, we propose a design framework that addresses both issues simultaneously: we frame the network design problem in a way that lets the designer tradeoff between modularity and mixing time, and we propose an an algorithm that can find extremal graphs under these criteria. Specifically, for the design framework, we take advantage of prior literature in the area of spectral graph theory and demonstrate how desirable aspects of organizational structure can be mapped parsimoniously onto the spectrum of the graph Laplacian derived from a matrix representation of that communication structure. Recent advances in convex optimization allow us to use elements of graph spectra in an objective function to be optimized. Finally, we also present and discuss the resulting communications structures that balance modularity and mixing time.

2 Spectral Theory Informs Design

Spectral graph theory (Cvetkovic and Sachs, 1998; Chung, 1997) is concerned with the relationships between the structure of a network and the eigenvalues, also called the "spectrum," of the matrix representation of the network. One major advantage of thinking of networks in terms of their spectra is that spectra are insensitive to permutations and labeling. All networks with the same structure have the same spectrum.[1] This property lets us

[1] The converse is not necessarily true, but is true with high probability.

avoid having to deal with the so-called "graph isomorphism problem," where many equivalent representations for structurally isomorphic graphs exist, making search and classification in graph space difficult. In essence, working with the spectra lets us focus on a more tractable and compact object, and one which corresponds to a unique graph with high probability (see section 2.3). Moreover, the values of the spectra provide enormously useful information about graph structure in a compact and accessible way. These properties make spectra the ideal mathematical objects to use in formalizing desiderata and constraints in network design problems.

In this paper, we adopt a particular design objective: we aim to design networks that both manifest distinct subgroups and yet are still "sufficiently connected". As we have seen in the previous section, these are well motivated goals. However, it is not obvious how to formalize them. Spectral theory gives us a means to frame this precisely. Existing work has not not examined such an objective; we provide:

- A spectral formalization of our modularity and mixing objective (section 2.2)
- A novel optimization problem based on this formulation that captures our design objective (section 3)
- An algorithm for approximately solving this problem (section 3.1)
- A set of numerical experiments based on this algorithm, and their results and interpretation.

2.1 Preliminaries

The standard matrix representation of a graph, where each entry represents the strength of the connection between the node indexed by the matrix row and column, is called the *adjacency* matrix. In this paper, we assume that each individual in the organization has equal communications capacity that they use fully. This implies that our matrix representations of the network must have rows and columns that can be normalized so that they all sum to 1 (such matrices are called "doubly stochastic"). Further, we assume that a given communication tie takes the same proportion of each connected individual's communication capacity. Together these properties imply that the matrix representation of the network must be symmetric about its diagonal.

Instead of working with the adjacency matrix, it can be useful to work with the graph *Laplacian* matrix given, for stochastic graphs, by $L = I - A$, where I is the identity matrix and A the adjacency matrix.[2] The spectrum of A and L are related but have distinct properties; those of the Laplacian match our needs and we consequently adopt it here. The matrix spectrum is simply the multiset of eigenvalues for the matrix, sorted in decreasing order of magnitude.[3] Such a spectrum can be plotted as a set of points, as illustrated in figure 1 and elaborated upon below.

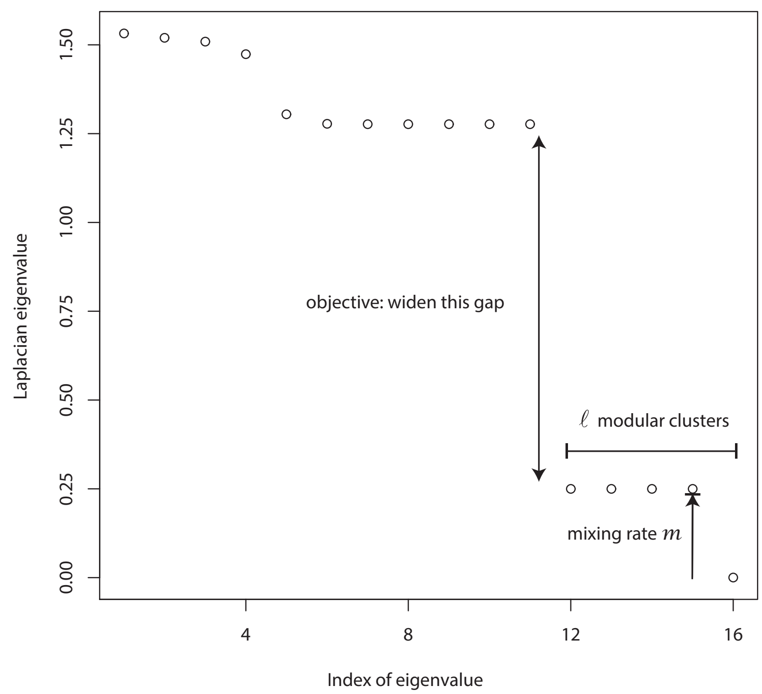

Figure 1: Illustration of the spectral framework, including objective and constraints

[2]In general the *Laplacian* is given by $L = D - A$, where D is the degree matrix, constructed by putting the row sums of A on the diagonal, with zeros elsewhere.

[3]The eigenvalues of a matrix M are given by $\{\lambda | M\boldsymbol{v} = \lambda \boldsymbol{v}, \boldsymbol{v} \neq 0\}$. The $\boldsymbol{v}$ are called the eigenvectors of the matrix: those vectors that when multiplied by the matrix yield a scaled copy of themselves. Each scale factor is a corresponding eigenvalue, λ.

2.2 The Laplacian Spectrum and Network Structure

The relative magnitude of the various spectral values correspond to specific structural properties of the corresponding network. We describe those necessary for capturing our design objective below.

2.2.1 Bounding the mixing time with m

The magnitude of the smallest Laplacian eigenvalue (hereafter, just "eigenvalue" for brevity) is always zero, and therefore of little immediate interest. However, the magnitude of the second smallest eigenvalue is also the graph's "algebraic connectivity" (Fiedler 1973) and is inversely related to the mixing time for Markov chains (Mohar 1997). In short, the larger the second smallest eigenvalue, the faster we expect information to diffuse through the network (Donetti et al. 2006). Because of its known connection to mixing time, we refer to the magnitude of the second smallest eigenvalue as m (see Figure 1). By tuning m a network designer has a spectral method for formalizing the idea of "sufficiently connected:" the larger the m, the more rapidly that communication structure is expected to diffuse information. However, raising m may come at the cost of other desirable features, such as the amount of modularity that is manifest in the network, as we shall see shortly.

2.2.2 Setting the number of modular clusters with ℓ

It is well known that the number of connected components of an undirected graph is equal to the number eigenvalues of the Laplacian that are equal to zero (Brouwer and Haemers 2011). For example, if there were four totally disconnected components, there would be four eigenvalues equal to zero. If, however, there existed weak connections among those distinct communities such that they are no longer disconnected components but rather modular clusters, then rather than having one zero for each cluster, we would have one small eigenvalue for each module (Donetti et al. 2006). Consequently, for a graph consisting of four modular clusters that are weakly connected to each other, the spectrum of the Laplacian (hereafter "spectrum") would contain four small eigenvalues, one of which would be zero (as there would be one component, and thus one eigenvalue equal to zero).

From the design point of view, then, we observe that if one desires a communication network with some number, ℓ, distinct modular clusters, then one should construct a graph with a spectrum containing ℓ small eigenvalues, one of which is zero (see Figure 1).

2.2.3 The rest of the spectrum

We have just argued that we want $\lambda_k, k \leq \ell$ to be small. But small relative to what? To make λ_ℓ relatively small, we need $\lambda_{\ell+1}$ to be large, and this in turn will drive up all $\lambda_k, \ell < k \leq n$, giving us a graph that is as modular as possible. A result provided by Newman[4] enables us to interpret this more clearly (Newman 2000):

$$\lambda_k(G^C) = n - \lambda_{n+2-k}(G) \text{ for } 2 \leq k \leq n \tag{1}$$

Where G is a graph and G^C is its complement. This result states that the kth largest eigenvalue is equivalent to the $k-1$ smallest eigenvalue of the complementary graph. So by driving the large eigenvalues up, we are driving down the small eigenvalues of the complementary graph. Thus, will make the complementary graph have a long mixing time (via $\lambda_2(G^C) = \lambda_n(G)$), and more broadly have the maximum number $(n - \ell)$ of largely disconnected modules. When n is large relative to ℓ this implies a largely disconnected complementary graph, and therefore a (primary) graph that is highly connected within its ℓ modules.

2.3 Co-spectral graphs

It is one thing to calculate the spectrum of a known graph and quite another to construct a graph with a given spectrum. Since we are using spectral properties to design networks with desirable structural properties, we are more concerned with the latter problem. The next section details our method for constructing matrices with desirable spectral properties.

[4]Attributed by Newman to Kel'mans.

Before we do so, however, we must take note of the issue of co-spectral graphs, or graphs with the same spectrum (Harary et al. 1971, Godsil and McKay 1982).

Although at present relatively little is known about which graphs have co-spectral partners (Van Dam and Haemers 2003), we do not believe this presents a substantial impediment to the present undertaking. Most fundamentally, we are presenting a framework for designing communication networks with properties that have spectral correlates. If by chance we construct a graph for which there exists a co-spectral partner that we do not find, we have still achieved our design goal, because co-spectral graphs have similar structure with respect to the features captured in the spectrum.

Additionally, but less essentially, enumerations of unweighted graphs that are co-spectral with respect to their Laplacian (Haemers and Spence 2004, Brouwer and Spence 2009, Cvetković 2012) show that the proportion of graphs with co-spectral partners is highest at $n = 9$ and decreases as n and the number of edges increase. Halbeisen and Hungerbühler (2000) show that for weighted graphs — which we employ here — there are almost surely no co-spectral partners. Therefore, we assert that by constructing weighted networks according to spectral parameters, we are not leaving anything important to our aims on the table.

3 Methods

Spectral theory has given us the means to formalize both of our design objectives:

- Sufficient connectivity, by imposing a lower bound, m, on the second smallest eigevnvalue λ_2, which ensures a fast enough mixing time.
- Modularity with ℓ clusters, by having ℓ small eigenvalues and $n - \ell$ large eigenvalues.

Our network design problem can then be cast the following non-linear optimization problem:

$$\max_{\boldsymbol{W}} \quad \lambda_{\ell+1}(\boldsymbol{W}) - \lambda_{\ell}(\boldsymbol{W}) \tag{2}$$

$$\text{s.t.} \quad \lambda_2 \geq m \tag{3}$$

$$\sum_j \boldsymbol{W}_{ij} = 1 \; \forall \, i \tag{4}$$

$$\boldsymbol{W}_{ij} = \boldsymbol{W}_{ji} \; \forall \, i, j \tag{5}$$

The objective, equation 2, maximizes the difference between the $\ell+1$ and ℓ Laplacian eigenvalue. The constraint, equation 3, ensures that the mixing time is at least m. Constraints 4 and 5 ensure stochasticity and symmetry respectively. Note that the variables in this formulation are the weights of matrix $\boldsymbol{W}$

3.1 Optimization Algorithm

The "eigenvalue problem," that of computing the eigenvalues for a known matrix, can be calculated in closed form for small matrices, and for large matrices by numerical algorithms such as QR that have been known since the early sixties (Francis 1961, 1962). However, "inverse eigenvalue problems," those of finding the graph that corresponds to a specific spectrum or specific spectral characteristics have proven vastly harder to solve (Chu 1998). Most admit no computationally tractable algorithm for obtaining a globally optimal solution.

Our formulation falls within this hard class of problem, and thus the best we can hope for is a high-quality approximation algorithm. We are not aware of any existing work that has looked at solving our particular spectral objective and constraints. We have therefore constructed our own approximation method by leveraging recent advances in Semi-Definite Programming (SDP) and Difference in Convex (DC) programming, which we next describe.

3.1.1 Semi-Definite Programming

Semi-Definite Programming (SDP) is a type of convex optimization that operates over a matrix variable, instead of the scalar variables seen in other convex optimization methods (Vandenberghe and Boyd 1996). SDP objectives are specified as the inner-product of the matrix variable, with a user-specified constant matrix. Similarly, SDP constraints consist of a bound on the inner-product between the matrix variable and another user-specified constant matrix. The minimal value for the objective is found, where the matrix variable is drawn from the cone of semi-definite matrices. Many problems can be cast into this structure, and because the resulting formulation is convex, it can be solved efficiently by, for example, interior point methods (Todd 2001, Wolkowicz et al. 2000, Alizadeh 1995).

For the present work, the key value of SDP is its ability to capture the sum of the k smallest Laplacian eigenvalues, S_k, as a concave function. Boyd et. al. have used this capability to solve certain Laplacian inverse eigenvalue problems directly (Boyd et al. 2004, Boyd 2006). For example, they formulate S_2 as a concave function which they can then maximize to solve for the Markov process with the fastest mixing time extremely efficiently. We leverage their result by moving their objective formulation to a constraint, obtaining a convex form for equation 3. Further, as the remaining constraints are linear, only our objective 2 fails to be directly representable as an SDP, which we address next.

First, suppose instead of focusing on λ_ℓ and $\lambda_{\ell+1}$, we adopted the more easily formulated objective of $max T_k - S_k$ (where T_k is the sum of the k largest Laplacian eigenvalues, a convex function). We observe that this is maximizing (and not minimizing) a convex function and thus fails to be a convex optimization. Consequently, our attempt to create a "maximal gap" by driving down small eigenvalues and up large ones is not a convex objective, and standard SDP will not alone be sufficient for solving the problem.

Moreover, even if this were convex, the solution found might not actually create a gap between λ_ℓ and $\lambda_{\ell+1}$, as the summations enable solutions with no gap. As a consequence, we really do need a way to capture λ_ℓ and $\lambda_{\ell+1}$. We can do this, by noting that $\lambda_\ell = S_\ell - S_{\ell-1}$

and $\lambda_{\ell+1} = S_{\ell+1} - S_\ell$. And thus

$$\lambda_{\ell+1} - \lambda_\ell = (S_{\ell+1} - S_\ell) - (S_\ell - S_{\ell-1}) = S_{\ell-1} + S_{\ell+1} - 2S_\ell \tag{6}$$

This objective captures our intent, and can be formulated by known SDP-style expressions. However, it can not be directly solved because the third term is non-convex.

3.1.2 Difference in Convex Programming

As we have seen, the formulation of equation 2 given in equation 6, is almost convex and solvable as and SDP, but not quite. Consequently, we are not going to be able to directly use convex optimization, and the best we can hope for is an approximately optimal algorithm. However, equation 6 is a *difference of convex functions* and, as such, is amenable to an algorithm known as the Concave-Convex Procedure (Yuille et al. 2002, Yuille and Rangarajan 2003). This is an iterative method for obtaining approximate solutions to problems with structure that has good convergence properties (Sriperumbudur and Lanckriet 2009). Our approach is to implement the Concave-Convex Procedure over our SDP program as follows:

We start with a random initial graph $\hat{\boldsymbol{W}}$. We then form a first-order Taylor expansion of the concave portion of the objective around $\hat{\boldsymbol{W}}$. Using this linear form, we can then approximate the objective as:

$$S_{\ell-1}(\boldsymbol{W}) + S_{\ell+1}(\boldsymbol{W}) - 2\left(S_\ell(\hat{\boldsymbol{W}}) + (\boldsymbol{W} - \hat{\boldsymbol{W}})S'_\ell(\hat{\boldsymbol{W}})\right) \tag{7}$$

This then, is directly solvable as an SDP, which we obtain using the CVX package (Grant and Boyd 2012, 2008). We then set $\hat{\boldsymbol{W}} \leftarrow \boldsymbol{W}$ and repeat until convergence.

3.2 Bounding the Objective Value

Because our optimization algorithms may be only locally optimal, it is useful to have a theoretical upper bound on the objective value in equation 2. If our numerical calculations

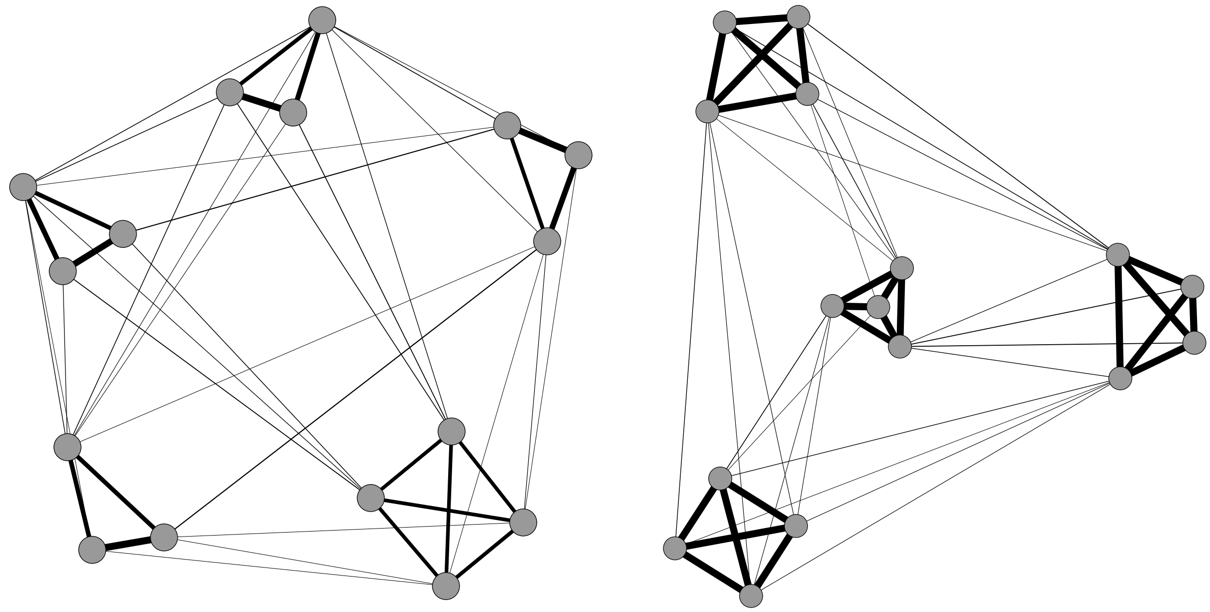

Figure 2: Two examples of sixteen person communication networks produced by the spectral design framework. The left hand network has $\ell = 5$ and the right hand $\ell = 4$ teams, both with mixing rate $m = 0.25$

are close to this value, we know we are optimal[5] To do this, we note that the sum of the Laplacian eigenvalues cannot exceed n (Chung 1996). Further we know that $\lambda_1 = 0$ and, from constraint 3, that the $\lambda_k \geq m, 2 \leq k \leq \ell$. Putting these together, we obtain that the sum of the largest eigenvalues is bound from above as $\sum_{\ell+1 \leq k \leq n} \lambda_k \leq n - m(\ell - 1)$. The smallest of these is made maximal at this bound and when each of these eigenvalues is of equal size, giving it a value of $\frac{n-m(\ell-1)}{\ell-n}$. The overall objective is then maximized when $\lambda_\ell = m$ exactly. Subtracting the upper bound on $\lambda_{\ell+1}$ from the lower bound on λ_ℓ, gives us an overall upper bound on the objective of: $\frac{n-m(\ell-1)}{n-\ell} - m$.

4 Experiment and Results

4.1 General Nature of Communication Networks Produced by the Spectral Design Framework

Figure 2 shows two examples of networks produced by our framework. Several features are immediately apparent. As expected, these networks have a clear modular structure, with strong intra-team connections. Additionally, there are weak ties connecting the teams in patterns that appear in the visualization as "fans." Intuitively, one could think of these fans as ties from one representative of a team to (usually) all the members of another team. Although such an organization structure is reminiscent of the "matrix organization" (Galbraith 1971), we are not aware any appearance of similar graphs in the literature on network structure *per se*.

The disposition of the fans has a definite structure, suggestive of a hierarchical "spiral" in the visualizations. In the right hand side of Figure 2, the central team has three "outgoing" fans; the team to the right has two; the team at the top has one; and the team at the bottom has none. A minority of the weak ties are not part of a fan structure: in this network, the most visually central individual has singleton ties to two individuals in other teams.

Results for 32-person networks are similar to those for 16-person networks. Figure 3 shows a network comparable to the 16-person network on the right-hand side of Figure 2, displaying the same hierarchical spiraling structure.

4.2 Distribution of Link Weights

Figure 4 gives an illustrative histogram of the link weights produced by our algorithm. In addition to the high density of strongest, within-team links, there are groups of ties with weights of decreasing orders of magnitude and less apparent structure. For the purposes of visualization, we omit ties below the first wave of weak ties. In the histogram, this is at $\ln(0.018) \approx -4$. Manual checks show that these very weak links do contribute to the

[5] However, the converse does not necessarily follow.

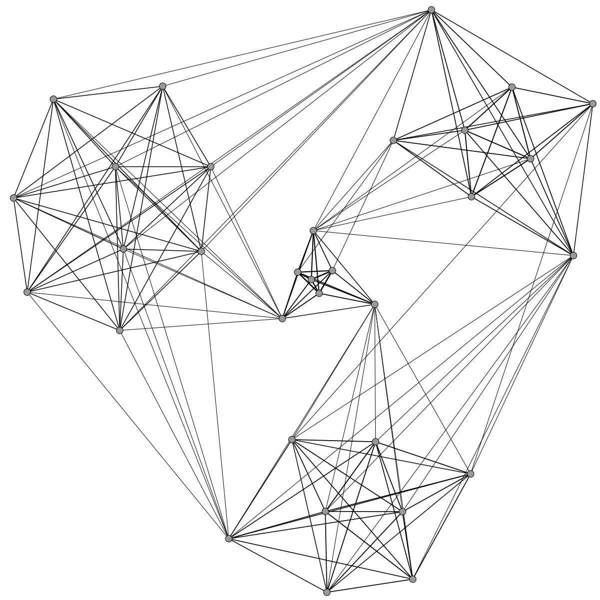

Figure 3: A 32-person network with $\ell = 4$ and $m = 0.15$

magnitude of the ℓth eigenvalue, and that removing them drops the ℓth eigenvalue below m. Nonetheless, for the purposes of legibility, they are omitted from the visualizations we present here.

4.3 Optimality of results

The spectral framework does not guarantee a specific number of individuals in each team or particular distribution of link weights. Our experience suggests that the experimental

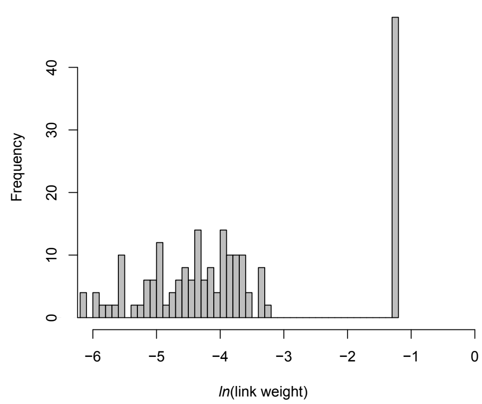

Figure 4: Histogram of ln(link weights).

runs that achieve a higher objective value also have a more equally sized teams and a more easily interpretable pattern of edges.

Table 1 gives the best objective values we achieved, divided by the theoretical upper bound calculated for each set of parameter values, as described in section 3.2. Each data point represents the best of 2000 independent runs of our algorithm, obtained by running each in parallel on a 1000 node computational cluster. Each run of the algorithm generally completes in less then 2 hours of CPU time on modern Xeon-class hardware. From the table we see that our algorithm is finding answers that are very near our bound in most of the cases. Where some provable optimality gap remains, it is unknown if this is due to the bound being loose, or the algorithm failing to find a sufficiently global optimum.

Table 1: Degree of optimality achieved

n	ℓ	m	optimality
16	6	0.15	0.938
16	6	0.20	0.938
16	6	0.25	0.938
16	5	0.15	0.917
16	5	0.20	0.919
16	5	0.25	0.922
16	4	0.15	1.000
16	4	0.20	1.000
16	4	0.25	0.999
32	8	0.15	0.937
32	8	0.20	0.939
32	6	0.20	0.975
32	4	0.15	0.984
32	4	0.20	0.978

5 Discussion

We have drawn connections between spectral graph theory and work on network structure and problem solving. The benefits of network modularity has been been well documented in the theory of networked problem solving, above and beyond the well-explored benefits of short average path lengths between all members of an organization. However, prior work on network design has not incorporated these insights from the theory on networked problem solving. Our spectral approach enables us to marry the networked problem solving theory with a network design methodology, and with an elegant formalism.

The communication networks generated by our optimization algorithm show novel structural features. In particular, the hierarchical "spirals" of fans of weak ties — connecting a single individual in one team to multiple members of another — allow inter-team connectivity, while maintaining a high degree of modularity. That said, the communication structures presented here are finely articulated and might be difficult to achieve in a real organization. However, this need not be difficult on a computer mediated crowdsourcing platform. An obvious implementation of tie strength would be as a fraction of the problem solving time spent "together," with the opportunity to exchange ideas or observe the progress of others.

By specifying a spectral interpretation of core network problem solving properties, like connectivity and modularity, we hope we have opened doors for future research on the design of problem-solving networks. Future computational experiments could extend this framework to networks of individuals with different communications capacities, thereby dropping the requirement that networks be representable by doubly-stochastic graphs. In addition to analytical and computational research, these results should be tested experimentally with human problem solvers to assess their performance (much as Shore et al. (2013) did with network structures received from prior literature on social networks).

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